Problem Set 10 – Relativity (`G0I36A`)

24/11/2020

1 Conservation of the Stress-Energy Tensor

In this problem, we will prove (in as much generality as we can) that the stress-energy tensor is always conserved.

1.) Let $g_{\mu\nu}(x)$ be the metric of our spacetime with coordinates x^μ. Now, consider doing an *infinitesimal* coordinate transformation of the form

$$x^\mu \to y^\mu = x^\mu + \xi^\mu(x) \ , \tag{1.1}$$

where y^μ are the new coordinates and $\xi^\mu(x)$ is some small, possibly coordinate-dependent shift. We will denote the metric in this new coordinate frame by $\tilde{g}_{\mu\nu}(y)$.

Since the metric is a tensor, its transformation under this coordinate transformation by: (see e.g. equation (2.30) in Carroll)

$$\tilde{g}_{\mu\nu}(y) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\sigma}{\partial y^\nu} g_{\rho\sigma}(x) \ . \tag{1.2}$$

Prove that this can be written explicitly as

$$\tilde{g}_{\mu\nu}(y) = g_{\mu\nu}(x) - g_{\nu\rho}(x)\partial_\mu \xi^\rho(x) - g_{\mu\sigma}(x)\partial_\nu \xi^\sigma(x) + \mathcal{O}(\xi^2) \ . \tag{1.3}$$

2.) Do a Taylor expansion of $\tilde{g}_{\mu\nu}(y)$ to show that it can be written as

$$\tilde{g}_{\mu\nu}(y) = \tilde{g}_{\mu\nu}(x) + \xi^\rho(x)\partial_\rho g_{\mu\nu}(x) + \mathcal{O}(\xi^2) \ . \tag{1.4}$$

3.) Combine your results from problems 2 and 3 to prove that, keeping only the lowest-order terms in the infinitesimal coordinate shift ξ^μ,

$$\delta g_{\mu\nu} = -\xi^\rho \partial_\rho g_{\mu\nu} - \partial_\mu \xi_\nu + \xi^\rho \partial_\mu g_{\nu\rho} - \partial_\nu \xi_\mu + \xi^\rho \partial_\nu g_{\mu\rho} \ , \tag{1.5}$$

where $\delta g_{\mu\nu}(x) \equiv \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x)$, and all terms in the above equation are evaluated at x.

4.) Use the definition of the covariant derivative to hence prove that under any infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$, the metric transforms as

$$\delta g_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \ . \tag{1.6}$$

Then, prove that this also implies that the inverse metric transforms as

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu \ . \tag{1.7}$$

5.) Argue that the principles of general relativity, and in particular general covariance and the equivalence principle, demand that the action S_M for the matter fields in our theory must be invariant under coordinate transformations.

6.) We define the stress-energy tensor from the matter action S_M by saying that under any small variation $\delta g_{\mu\nu}$ of the metric, the matter action transforms as

$$\delta S_M = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu}\ . \tag{1.8}$$

Prove that, in order for S_M to be invariant under coordinate transformations, we must have

$$\int d^4x\,\sqrt{-g}\,T^{\mu\nu}\nabla_\mu\xi_\nu = 0\ , \tag{1.9}$$

for any infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$.

7.) Use integration by parts to show that this in turn means that

$$\int d^4x\,\sqrt{-g}\,\left(\nabla_\mu T^{\mu\nu}\right)\xi_\nu = 0\ , \tag{1.10}$$

up to some boundary terms that we will ignore for now.

8.) At this point, the only thing we have assumed about ξ^μ is that it is small. It can still be any arbitrary function of the spacetime coordinates. Argue that the only way (1.10) can vanish for arbitrary ξ^μ is to require that

$$\nabla_\mu T^{\mu\nu} = 0\ . \tag{1.11}$$

Thus, general covariance of our theory requires conservation of the stress-energy tensor.

9.) Use this result to prove the following corollary:

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R\ . \tag{1.12}$$

2 Orbits around Reissner-Nordström Black Holes

Consider the four-dimensional electric Reissner-Nordström black hole

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega^2\ , \tag{2.1}$$

where

$$\Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\ , \tag{2.2}$$

and we have set $c = 1$. In this problem, we will try to understand the orbital trajectories of (neutral) particles (both massless and massive) that follow geodesics around this black hole.

10.) What are the Killing vectors on this background? How many are there? What conserved quantity do each of these Killing vectors correspond to?

11.) Use two of these conserved quantities to argue that we can, without loss of generality, orient our coordinate frame such that any single particle's path is fixed at $\theta = \frac{\pi}{2}$.

12.) Argue that the remaining two Killing vectors can be expressed as $K = \partial_t$ and $R = \partial_\phi$. Then, define the following quantities:

$$E = -K_\mu\frac{dx^\mu}{d\lambda}\ ,\quad L = R_\mu\frac{dx^\mu}{d\lambda}\ , \tag{2.3}$$

where $x^\mu(\lambda)$ denotes the trajectory of the geodesic the orbiting particle follows. What are these quantities physically? Are they conserved for the Reissner-Nordström black hole? Why or why not?

13.) As we have discussed before, the quantity

$$\epsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{2.4}$$

will also be conserved along the particle's trajectory. We will choose the normalization of our affine parameter λ such that ϵ takes values of only -1, 0, and $+1$. Which of these values corresponds to a massive particle? Which to a massless particle? What does the last remaining one correspond to?

14.) Compute ϵ for the Reissner-Nordström background, and show that it can be written as

$$\epsilon = \Delta \left(\frac{dt}{d\lambda} \right)^2 - \frac{1}{\Delta} \left(\frac{dr}{d\lambda} \right)^2 - r^2 \left(\frac{d\phi}{d\lambda} \right)^2 \ . \tag{2.5}$$

15.) Use your results from problem 12 to eliminate $\frac{dt}{d\lambda}$ and $\frac{d\phi}{d\lambda}$ in the above equation. You should be left with a differential equation that depends only on r and some constants. In particular, show that this differential equation takes the form

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 + V(r) = \mathcal{E} \ , \tag{2.6}$$

where $V(r)$ and $\mathcal{E}$ are given by

$$V(r) = \frac{1}{2}\epsilon - \epsilon \frac{GM}{r} + \frac{L^2 + \epsilon G Q^2}{2r^2} - \frac{GML^2}{r^3} + \frac{GL^2Q^2}{2r^4} \ , \tag{2.7}$$

$$\mathcal{E} = \frac{1}{2} E^2 \ . \tag{2.8}$$

This differential equation (2.6) is exactly the equation for a classical particle of "energy" $\mathcal{E}$ moving in a one-dimensional potential $V(r)$. Therefore, we can analyze the radial motion of geodesic orbits just by doing a very classical analysis of this equation.

16.) Argue that circular orbits are allowed only at particular values of r such that $V'(r) = 0$. Write down (but do not yet solve!) the condition for circular orbits.

17.) Consider a massless particle, with L, M, and Q all non-zero. Where are the allowed circular orbits? Are they stable or unstable?

18.) What happens to these circular orbits in the Schwarzschild limit where $Q \to 0$? What about the extremal limit where $Q^2 = GM^2$?

19.) Why have I not asked you to find all circular orbits for massive particles? Why might this be difficult to do?